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The construction of R^2 actions in $D = 4$, $N = 1$ supergravity

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Abstract. Actions containing R^2 terms in $d = 4$, $N = 1$ supergravity are constructed in the on-shell and the new minimal formulation of the theory. The basic feature in both cases is an analogy between supersymmetric Yang-Mills theory and supergravity. This analogy is also used to construct supersymmetric Lorentz Chern-Simons terms, which allows a derivation in $d = 4$ of an effective action resulting from string compactification.

1. Introduction

The interest in the construction and properties of higher order (super)gravity actions has increased considerably since the advent of superstring theories. This is because superstrings predict that the ultimate theory of quantum gravity, when expressed in field theoretical terms, involves a power series in the Riemann curvature tensor. The lowest order term (if the cosmological constant vanishes) in this series is the Einstein-Hilbert action, and higher order contributions are suppressed by inverse powers of the string tension. Therefore these higher order terms will play a minor role in very low energy physics, but they are likely to be essential in understanding the consistency of quantum gravity. Therefore it is of considerable interest to investigate properties of R^n theories in four-dimensional spacetime.

One scenario which makes contact between the heterotic superstring theory in ten dimensions and elementary particle physics in four dimensions runs as follows [1]. First an effective action for the massless fields of the string theory is constructed in ten dimensions. Its form can be determined, e.g. by comparison with string amplitude calculations. This effective action can be expanded in powers of α , where α is the inverse string tension. The lowest order, $O(\alpha^0)$, terms, correspond to ten-dimensional $N = 1$ supergravity coupled to Yang-Mills theory [2, 3]. The effective theory has, hopefully, a preferred compactification to four dimensions, leaving one ($N = 1$) supersymmetry unbroken. The feasibility of this scenario hinges on the presence of the $O(\alpha)$ Lorentz Chern-Simons term [4] in the ten-dimensional effective action.

Recently, a new method was employed to obtain the supersymmetric Lorentz Chern-Simons terms in $d = 10$ [5], and this was later extended to terms involving R^4 [6]. The essential step in this method is the fact that in certain supergravity theories a suitable combination of spin-connection and other fields transforms under supersymmetry exactly as the gauge field of the Yang-Mills multiplet. This observation was first used in a superconformal context in $d = 6$ [7].

In this paper we use this method to construct four-dimensional supersymmetric R^2 actions. Since we have in mind the application to string effective actions, we discuss not only the R^2 invariant itself, but also an invariant containing the Lorentz Chern-Simons term. This last result makes contact with an effective action derived by Witten

[8]. To establish this contact a duality transformation is required. It is remarkable and satisfying that our method provides an easy way to obtain this result.

A major difference between four- and ten-dimensional supergravity theories is that in the latter theory the supersymmetry algebra closes only on-shell, whereas in $d = 4$ one has, besides the on-shell version, a wide choice of off-shell formulations with differing sets of auxiliary fields. With higher derivative actions these different formulations are not equivalent. In this paper, we will apply the Yang–Mills analogy mentioned above to two cases in $d = 4$: the original formulation without auxiliary fields [9], and the new minimal auxiliary field formulation [10] of $N = 1$ supergravity.

The construction of the R^2 action in the new minimal formalism will give insight into our procedure, and indeed reproduces, in terms of component fields, the results already obtained by other methods [11–13]. Including the supersymmetrisation of Chern–Simons terms poses no essential problem.

The construction without auxiliary fields resembles more closely the situation in $d = 10$. In this case supersymmetry holds only order by order in α . Even in $O(\alpha)$ the $d = 4$ construction shows a new feature. The identification of the spin-connection with the Yang–Mills field holds only modulo field equations. As we shall see, this implies that the supersymmetry transformation of the gravitino field acquire $O(\alpha)$ modifications. It is interesting that, at this $O(\alpha)$ level, we can follow two different approaches to construct supersymmetric $R + R^2$ actions. The two methods, one of which can be systematically extended to higher orders in α , require different modifications to the supersymmetry transformation rules. Supersymmetric R^2 actions were previously considered using superconformal methods [14] and the old minimal [15] formulation of supergravity [16, 17].

In sections 2 and 3 we discuss mainly the invariant associated with the square of the Riemann tensor. In the effective action obtained from superstring theory, other terms are present as well, in particular the Lorentz Chern–Simons terms which are essential in the cancellation of anomalies in $d = 10$ [4]. In section 4 we will discuss the construction of such terms. Some conclusions are gathered in section 5.

2. New minimal supergravity

In four dimensions there are many different off-shell versions of supergravity. In the new minimal formulation the auxiliary fields are an antisymmetric tensor gauge field $B_{\mu\nu}$ and a vector gauge field V_μ , which gauges the chiral symmetry in the supergravity multiplet [10]. Formally, the two-index tensor gauge field gives a resemblance to the ten-dimensional supergravity multiplet, where of course this field is physical rather than auxiliary. Also in six dimensions such a field is present. Not surprisingly then, the method of [5, 6] for constructing R^2 actions is ideally suited for this formulation of the four-dimensional theory.

Before discussing the R^2 actions, let us recapitulate the main features of new minimal supergravity and its tensor calculus. The supergravity fields in the new minimal formalism are the bosons e_μ^a , $B_{\mu\nu}$, V_μ , and the gravitino ψ_μ . The supersymmetry and chiral transformations, with parameters ε and Λ respectively, are[†]

$$\begin{aligned}\delta e_\mu^a &= \frac{1}{2}\bar{\varepsilon}\gamma^a\psi_\mu & \delta\psi_\mu &= \mathcal{D}_\mu(\Omega_+, V_+)\varepsilon + i\gamma_5\Lambda\psi_\mu \\ \delta B_{\mu\nu} &= \frac{3}{2}\bar{\varepsilon}\gamma_{[\mu}\psi_{\nu]} & \delta V_\mu &= \frac{1}{8}i\bar{\varepsilon}\gamma_5\gamma_\mu\gamma^{ab}\psi_{ab} + \partial_\mu\Lambda\end{aligned}\quad (2.1)$$

[†] The conventions used in this paper are those of [18]. The antisymmetric product of gamma matrices is denoted by $\gamma^{a_1 a_2 \dots a_n} \equiv \gamma^{[a_1} \gamma^{a_2} \dots \gamma^{a_n]}$.

where we have defined the combinations

$$\Omega_{\mu\pm}^{ab} = \omega_{\mu}^{ab}(e, \psi) \pm \hat{H}_{\mu}^{ab} \quad V_{\mu+} = V_{\mu} + \frac{1}{6}i\epsilon_{\mu}^{abc}\hat{H}_{abc}. \quad (2.2)$$

The covariant curvatures are

$$\begin{aligned} \psi_{\mu\nu} &= \mathcal{D}_{\mu}(\Omega_{+}, V_{+})\psi_{\nu} - \mathcal{D}_{\nu}(\Omega_{+}, V_{+})\psi_{\mu} \\ \hat{H}_{\mu\nu\rho} &= \partial_{[\mu}B_{\nu\rho]} - \frac{3}{4}\bar{\psi}_{[\mu}\gamma_{\nu}\psi_{\rho]}. \end{aligned} \quad (2.3)$$

Derivatives \mathcal{D} are covariant with respect to Lorentz and chiral transformations. Thus the covariant derivative of the supersymmetry parameter ϵ in (2.1) is given by

$$\mathcal{D}_{\mu}(\Omega_{+}, V_{+})\epsilon \equiv (\partial_{\mu} - \frac{1}{4}\Omega_{\mu+}^{ab}\gamma_{ab} - i\gamma_5 V_{\mu+})\epsilon. \quad (2.4)$$

All hatted curvatures are supercovariant.

The above fields form an off-shell representation of supergravity. The commutator of two supersymmetry transformations with parameters ϵ_1 and ϵ_2 contains in the standard way field-dependent gauge transformations. In particular, there is a field-dependent Lorentz transformation with parameter $\Lambda^{ab} = -\frac{1}{2}\bar{\epsilon}_2\gamma^{\mu}\epsilon_1\Omega_{\mu-}^{ab}$, which therefore contains the field strength of the antisymmetric tensor gauge field $B_{\mu\nu}$.

To construct actions, it is necessary to introduce some of the results of tensor calculus in this new minimal formalism [19]. The basic multiplet is the general multiplet, with $8+8$ bosonic and fermionic degrees of freedom. The other multiplets that we will require, chiral, linear and vector multiplets, are submultiplets of this general multiplet. The supersymmetry and chiral transformation of the general multiplet coupled to supergravity are the following

$$\begin{aligned} \delta C &= \frac{1}{2}i\bar{\epsilon}\gamma_5\chi \\ \delta\chi &= \frac{1}{2}\gamma^a(v_a + i\gamma_5 D_a C)\epsilon + \frac{1}{2}(M + i\gamma_5 N)\epsilon - i\gamma_5\Lambda\chi \\ \delta M &= \frac{1}{2}\bar{\epsilon}(\lambda + \not{D}\chi - \frac{1}{12}\gamma^{abc}\chi\hat{H}_{abc}) + 2\Lambda N \\ \delta N &= \frac{1}{2}i\bar{\epsilon}\gamma_5(\lambda + \not{D}\chi - \frac{1}{12}\gamma^{abc}\chi\hat{H}_{abc}) - 2\Lambda M \\ \delta v_a &= \frac{1}{2}\bar{\epsilon}\gamma_a\lambda + \frac{1}{2}\bar{\epsilon}D_a\chi - \frac{1}{8}\bar{\epsilon}\gamma^{cd}\chi\hat{H}_{acd} - \frac{1}{12}\bar{\epsilon}\gamma_5\chi\epsilon_{abcd}\hat{H}^{bcd} \\ \delta\lambda &= -\frac{1}{2}\gamma^{ab}\epsilon D_a v_b + \frac{1}{2}i\gamma_5\epsilon D - \frac{1}{8}\gamma^{ab}\epsilon\bar{\psi}_{ab}\chi + i\gamma_5\Lambda\lambda \\ \delta D &= \frac{1}{2}i\bar{\epsilon}\gamma_5\not{D}\lambda + \frac{1}{8}i\bar{\epsilon}\gamma_5\gamma^{abc}\lambda\hat{H}_{abc}. \end{aligned} \quad (2.5)$$

The derivatives D in (2.5) are supercovariant, and contain covariantisations with ω_{μ}^{ab} and V_{μ} . Note that the $\chi\hat{H}$ terms in δM , δN and δv_a cannot all be absorbed by using $D_a(\Omega_{-}, V_{+})\chi$ instead of $D_a(\omega, V)\chi$. However, by a redefinition of λ with a $\chi\hat{H}$ term we could have achieved this in δM and δN (not in δv_a !). The formulation (2.5) has the advantage that the restriction $\lambda=0$ is allowed, leading to a chiral multiplet of weight zero.

The chiral multiplet contains $4+4$ degrees of freedom, in the fields $[A, B, \phi, F, G]$. It can be defined for arbitrary chiral weight n (and be generalised to arbitrary spin), leading to the following local supersymmetry and chiral transformations:

$$\begin{aligned} \delta A &= \frac{1}{2}\bar{\epsilon}\phi + n\Lambda B & \delta B &= \frac{1}{2}i\bar{\epsilon}\gamma_5\phi - n\Lambda A \\ \delta\phi &= \frac{1}{2}\gamma^a(D_a A + i\gamma_5 D_a B)\epsilon + \frac{1}{2}(F + i\gamma_5 G)\epsilon + i(n-1)\gamma_5\Lambda\phi \\ \delta F &= \frac{1}{2}\bar{\epsilon}\not{D}\phi - \frac{1}{24}\bar{\epsilon}\gamma^{abc}\phi\hat{H}_{abc} + \frac{1}{8}n\bar{\epsilon}(A + i\gamma_5 B)\gamma^{cd}\psi_{cd} - (n-2)\Lambda G \\ \delta G &= \frac{1}{2}i\bar{\epsilon}\gamma_5\not{D}\phi - \frac{1}{24}i\bar{\epsilon}\gamma_5\gamma^{abc}\phi\hat{H}_{abc} + \frac{1}{8}in\bar{\epsilon}\gamma_5(A + i\gamma_5 B)\gamma^{cd}\psi_{cd} + (n-2)\Lambda F. \end{aligned} \quad (2.6)$$

Thus the general multiplet contains a chiral multiplet of weight $n=2$, which starts with M and N , as well as a chiral multiplet of weight zero starting with v (such that $v_a = D_a v$) and C . This last multiplet is obtained by setting $\lambda = 0$ in (2.5).

The vector multiplet can be extracted from (2.5) by setting $C = \chi = M = N = 0$, and by defining $A_\mu \equiv e_\mu^a v_a$. The corresponding transformation rules are:

$$\begin{aligned}\delta A_\mu &= \frac{1}{2} \bar{\epsilon} \gamma_\mu \lambda \\ \delta \lambda &= -\frac{1}{4} \gamma^{ab} \epsilon \hat{F}_{ab}(A) + \frac{1}{2} i \gamma_5 \epsilon D + i \gamma_5 \Lambda \lambda \\ \delta D &= \frac{1}{2} i \bar{\epsilon} \gamma_5 \not{D}(\Omega_-) \lambda\end{aligned}\quad (2.7)$$

and can be generalised to the Yang-Mills case. This extension requires only that Yang-Mills covariantisations be present in \hat{F} and in all covariant derivatives.

Action formulas can be based on the F component of a chiral multiplet with weight two, or the D component of the general or vector multiplet (F type and D type, respectively). These action formulas, which can be derived using the Noether method, are:

$$\mathcal{L}_D = e \{ D + \frac{1}{2} i \bar{\psi}_\mu \gamma_5 \gamma^\mu \lambda + \frac{1}{3} i e^{-1} \epsilon^{\mu\nu\lambda\rho} v_\mu \partial_\nu B_{\lambda\rho} \} \quad (2.8)$$

$$\mathcal{L}_F = e \{ F + \frac{1}{2} \bar{\psi}_\mu \gamma^\mu \phi + \frac{1}{4} \bar{\psi}_\mu \gamma^{\mu\nu} (A + i \gamma_5 B) \psi_\nu \}. \quad (2.9)$$

The construction of the R^2 action relies on the fact that the supergravity multiplet (2.1) contains a non-Abelian vector multiplet. Its components are

$$[\Omega_\mu^{ab}, \psi^{ab}, -2\hat{F}^{ab}(V_+)] \quad (2.10)$$

and the gauge group is $SO(3, 1)$. We will discuss this non-Abelian multiplet in more detail below. In addition, there is an Abelian vectormultiplet with components

$$[V_\mu, -\frac{1}{4} i \gamma_5 \gamma^{ab} \psi_{ab}, -\frac{1}{4} (\hat{R}(\omega) + \hat{H}^2)] \quad (2.11)$$

where $\hat{R}(\omega)$ is the supercovariant Riemann scalar. Using the action formula (2.8) for this vector multiplet, one obtains the supergravity action. With the conventional normalisation, the result becomes

$$\mathcal{L}_{SG} = -\frac{1}{2} e (\hat{R}(\omega) + \hat{H}^2) - \frac{1}{4} e \bar{\psi}_\mu \gamma^\mu \gamma^{ab} \psi_{ab} + \frac{2}{3} i \epsilon^{\mu\nu\lambda\rho} V_\mu \partial_\nu B_{\lambda\rho} \quad (2.12)$$

$$= -\frac{1}{2} e \{ R(\omega) + \bar{\psi}_\mu \gamma^{\mu\nu\lambda} \mathcal{D}_\nu(\omega, V) \psi_\lambda + \hat{H}_{abc} \hat{H}^{abc} \} + \frac{2}{3} i \epsilon^{\mu\nu\lambda\rho} V_\mu \partial_\nu B_{\lambda\rho}. \quad (2.13)$$

The action for supersymmetric Yang-Mills theory coupled to supergravity is an F -type action, which is obtained from the chiral multiplet of weight two constructed with bilinears of the fields of the vector multiplet:

$$\begin{aligned}A &\rightarrow -\frac{1}{4} \bar{\lambda} \lambda & B &\rightarrow -\frac{1}{4} i \bar{\lambda} \gamma_5 \lambda \\ \phi &\rightarrow -\frac{1}{4} \gamma^{ab} \lambda \hat{F}_{ab} - \frac{1}{2} i \gamma_5 \lambda D \\ F &\rightarrow -\frac{1}{4} \hat{F}_{ab} \hat{F}^{ab} + \frac{1}{2} D^2 - \frac{1}{2} \bar{\lambda} \not{D} \lambda - \frac{1}{8} \bar{\lambda} \gamma^{abc} \lambda \hat{H}_{abc} \\ G &\rightarrow -\frac{1}{8} i \epsilon^{abcd} \hat{F}_{ab} \hat{F}_{cd} + \frac{1}{2} i \bar{\lambda} \gamma_5 \not{D} \lambda.\end{aligned}\quad (2.14)$$

The action formula (2.9), applied to the multiplet (2.14), gives the following Yang-Mills action:

$$\mathcal{L}_{YM} = -e \beta \operatorname{tr} \{ \frac{1}{4} \hat{F}_{ab} \hat{F}^{ab} - \frac{1}{2} D^2 + \frac{1}{2} \bar{\lambda} \not{D} \lambda + \frac{1}{8} \bar{\lambda} \gamma^{abc} \lambda \hat{H}_{abc} \\ + \frac{1}{2} \bar{\psi}_\mu \gamma^\mu (\frac{1}{4} \gamma^{ab} \lambda \hat{F}_{ab} + \frac{1}{2} i \gamma_5 \lambda D) + \frac{1}{16} \bar{\psi}_\mu \gamma^{\mu\nu} (\bar{\lambda} \lambda - \gamma_5 \bar{\lambda} \gamma_5 \lambda) \psi_\nu \} \quad (2.15)$$

$$\begin{aligned}&= -e \beta \operatorname{tr} \{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D^2 + \frac{1}{2} \bar{\lambda} \not{D}(\omega, V, A) \lambda \\ &\quad + \frac{1}{8} \bar{\psi}_\mu \gamma^{ab} \gamma^\mu \lambda (F_{ab} + \hat{F}_{ab}) + \frac{1}{8} \bar{\lambda} \gamma^{abc} \lambda \hat{H}_{abc} \}.\end{aligned}\quad (2.16)$$

Here β is the inverse squared of the Yang-Mills coupling constant.

Let us now discuss in more detail the $SO(3, 1)$ Yang-Mills multiplet (2.10). In the new minimal formalism the supersymmetry transformation of the spin-connection $\omega_\mu^{ab}(e, \psi)$ reads

$$\delta\omega_\mu^{ab}(e, \psi) = \frac{1}{2}\bar{\epsilon}\gamma_\mu\psi^{ab} - \frac{3}{4}\bar{\epsilon}\gamma^{[a}\psi^{bc]}e_{\mu c} + \frac{1}{2}\bar{\epsilon}\gamma_c\psi_\mu\hat{H}^{abc}. \quad (2.17)$$

Clearly this is not the transformation rule of the first component of a Yang-Mills multiplet (see (2.7)). However, we can use the fact that

$$\delta\hat{H}_{abc} = -\frac{3}{4}\bar{\epsilon}\gamma_{[a}\psi_{bc]} \quad (2.18)$$

to construct a combination of ω and \hat{H} that does have this property. Clearly the correct combination is Ω_μ^{ab} . As we mentioned before, the commutator of two supersymmetry transformations contains an Ω_- -dependent Lorentz transformation. When the Yang-Mills multiplet is coupled to supergravity, the algebra also contains a Yang-Mills transformation with parameter $\Lambda = -\frac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1 A_\mu$, so that we see that the Yang-Mills vector field and Ω_- appear on the same footing. It is straightforward to verify that indeed the transformation of Ω_- correctly leads to a Yang-Mills multiplet. The transformations are:

$$\begin{aligned} \delta\Omega_\mu^{ab} &= \frac{1}{2}\bar{\epsilon}\gamma_\mu\psi^{ab} \\ \delta\psi^{ab} &= -\frac{1}{4}\gamma^{cd}\epsilon\hat{R}_{cd}^{ab}(\Omega_-) - i\gamma_5\epsilon\hat{F}^{ab}(V_+) \\ \delta\hat{F}^{ab}(V_+) &= -\frac{1}{4}i\bar{\epsilon}\gamma_5\mathcal{D}(\Omega_-)\psi^{ab} \end{aligned} \quad (2.19)$$

where now the Ω_- covariantisation in the last line acts both on the fermion structure of ψ^{ab} and on the $SO(3, 1)$ indices ab . The derivation of the transformation rule of ψ^{ab} makes use of the following identity for Riemann tensors with torsion:

$$\hat{R}_{abcd}(\Omega_+) = \hat{R}_{cdab}(\Omega_-). \quad (2.20)$$

The proof of (2.20) requires the Bianchi identity of \hat{H} , which reads $D_{[a}(\omega)\hat{H}_{bcd]} = 0$.

Therefore the construction of an R^2 -action has become a simple matter. We have to use the general Yang-Mills invariant (2.15) or (2.16) for the multiplet (2.19). The resulting action is invariant under the local supersymmetry transformations (2.1), and the transformations (2.19), which of course follow from (2.1). So we find (using (2.16)):

$$\begin{aligned} \mathcal{L}_2 &= -e\alpha\{\frac{1}{4}R_{\mu\nu}^{ab}(\Omega_-)R^{\mu\nu ab}(\Omega_-) - 2\hat{F}_{ab}(V_+)\hat{F}^{ab}(V_+) + \frac{1}{2}\bar{\psi}^{ab}\mathcal{D}(\omega, V, \Omega_-)\psi_{ab} \\ &\quad + \frac{1}{8}\bar{\psi}_\mu\gamma^{cd}\gamma^\mu\psi_{ab}(R_{cd}^{ab}(\Omega_-) + \hat{R}_{cd}^{ab}(\Omega_-)) + \frac{1}{8}\bar{\psi}^{ab}\gamma^{\mu\nu\lambda}\psi_{ab}\hat{H}_{\mu\nu\lambda}\}. \end{aligned} \quad (2.21)$$

We have multiplied by an arbitrary coupling constant α , analogous to β in the Yang-Mills case.

Since we are working with an off-shell formulation of supergravity, there are no modifications to the supersymmetry transformation rules of the supergravity fields due to the coupling to Yang-Mills. Therefore (2.21) is *exactly* supersymmetric, and not only to order α . This in contradistinction to $d=10$, where no (non-linear) off-shell formulation of the supergravity and Yang-Mills multiplets is available. Another difference with $d=10$ is of course that there the field $B_{\mu\nu}$ is a physical field. Here, at least on the level of the R action, the field $B_{\mu\nu}$ is an auxiliary field and can be eliminated from the theory. This is no longer true in (2.21), since higher powers as well as derivatives of \hat{H} appear. Therefore $B_{\mu\nu}$ becomes a physical field. Still another difference is that in four dimensions the R^2 invariant (2.21), containing the Riemann tensor, can be expressed in terms of combinations of Ricci tensors and scalars using the super-Gauss-Bonnet theorem [16].

Before turning to on-shell supergravity we briefly indicate the construction of other R^2 actions besides the one derived in (2.21).

An action containing the square of the curvature scalar can be constructed as follows. We start from the Abelian vector multiplet (2.11), and form a chiral multiplet using relations (2.14). Then the F -type action (2.9) yields a gravitational action containing an $(\hat{R}(\omega) + \hat{H}^2)^2$ term. The bosonic part of this action reads:

$$\mathcal{L}_{3\text{ bos}} = -\frac{1}{4}e\alpha\{F_{ab}(V)F^{ab}(V) - \frac{1}{8}(R(\omega) + H^2)^2\}. \quad (2.22)$$

Finally an action that involves the square of the Ricci tensor is obtained as follows [19]. We start from a linear multiplet $[C, \chi, v_a]$ coupled to new minimal supergravity. This is obtained by setting the chiral submultiplet of weight two contained in the general multiplet (2.5) equal to zero. The remaining fields $[C, \chi, v_a]$ transform according to

$$\begin{aligned} \delta C &= \frac{1}{2}i\bar{\epsilon}\gamma_5\chi \\ \delta\chi &= \frac{1}{2}\gamma^a(v_a + i\gamma_5 D_a C)\epsilon \\ \delta v_a &= -\frac{1}{2}\bar{\epsilon}\gamma_{ab}D^b\chi - \frac{1}{24}\bar{\epsilon}\gamma_{abcd}\chi\hat{H}^{bcd}. \end{aligned} \quad (2.23)$$

The vector field v_a must satisfy the constraint

$$D^a v_a - \frac{1}{3}i\epsilon_{abcd}(D^a C)\hat{H}^{bcd} + \frac{1}{4}\bar{\chi}\gamma_{ab}\psi^{ab} = 0 \quad (2.24)$$

which is the local version of the transversality condition $\partial_\mu v^\mu = 0$, holding in the rigid case. From a linear multiplet we can form a general multiplet with first component equal to C^2 . The D -component field is the following expression in terms of the fields of the linear multiplet $[C, \chi, v_a]$:

$$\begin{aligned} D = & -(v_a)^2 - (D_a C)^2 - 2(D_a D^a C)C + \bar{\chi}\not{D}\chi \\ & - \frac{2}{3}i\epsilon_{abcd}v^a\hat{H}^{bcd}C - \frac{1}{12}\bar{\chi}\gamma^{abc}\chi\hat{H}_{abc} - \frac{1}{2}i\bar{\chi}\gamma_5\gamma_{ab}\psi^{ab}C. \end{aligned} \quad (2.25)$$

Then using the D -type action (2.8) it is possible to write down an action for the fields of a linear multiplet.

The construction of an action containing the square of the Ricci-tensor is based on the observation that

$$[\frac{1}{6}\epsilon_{abcd}\hat{H}^{bcd}, \frac{1}{4}i\gamma_{abc}\psi^{bc}, -\frac{1}{2}i\hat{E}_{ab}(\Omega_-) - \frac{1}{2}\epsilon_{abcd}\hat{F}^{cd}(V_+)] \quad (2.26)$$

where

$$\hat{E}_{ab}(\Omega_-) \equiv \hat{R}_{ab}(\Omega_-) - \frac{1}{2}\delta_{ab}\hat{R}(\Omega_-) \quad (2.27)$$

is the supercovariant Einstein tensor, transforms as a linear multiplet with an additional Lorentz index. Using (2.25) and the action (2.8) for the multiplet (2.26) we find an action that involves the square of the Ricci tensor. The bosonic part of this action is given by:

$$\begin{aligned} \mathcal{L}_{4\text{ bos}} = & e\alpha\{\frac{1}{4}R_{ab}(\Omega_-)R^{ab}(\Omega_-) - \frac{1}{2}i\epsilon^{abcd}\mathcal{D}_e H_{ab}^e F_{cd}(V_+) \\ & - F_{ab}(V_+)F^{ab}(V_+) - \frac{1}{6}\mathcal{D}_a H_{bcd}\mathcal{D}^a H^{bcd}\}. \end{aligned} \quad (2.28)$$

It is now possible to combine the actions (2.21), (2.22) and (2.28) in such a way that the Gauss-Bonnet combination of the R^2 terms,

$$\mathcal{L}_{\text{GB}} = -e\alpha\{R_{\mu\nu}{}^{ab}(\omega)R^{\mu\nu ab}(\omega) - 4R_{\mu\nu}(\omega)R^{\mu\nu}(\omega) + R(\omega)R(\omega)\} \quad (2.29)$$

arises. It is then straightforward to verify that in such a combination the auxiliary fields V_μ , $B_{\mu\nu}$ of new minimal supergravity do not contribute. In this way one can check the validity of the super-Gauss-Bonnet theorem, i.e. the fact that the supersymmetric extension of (2.29) transforms into a total derivative, in new minimal supergravity.

3. On-shell supergravity

In this section we will discuss a similar construction of the R^2 action in $d=4$ using the formulation of supergravity with physical fields (vierbein and gravitino) only. We shall see that the use of the Yang-Mills analogy is still helpful for the construction of higher order supergravity actions.

In the absence of auxiliary fields the supergravity multiplet [9] has the following transformation rules:

$$\delta e_\mu{}^a = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \quad \delta\psi_\mu = \mathcal{D}_\mu(\omega)e. \quad (3.1)$$

The commutator of two supersymmetry transformations now closes modulo the equations of motion, which follow from the supergravity action

$$\mathcal{L}_{\text{SG}} = -\frac{1}{2}e\{R(\omega) + \bar{\psi}_\mu\gamma^{\mu\nu\lambda}\mathcal{D}_\nu(\omega)\psi_\lambda\}. \quad (3.2)$$

These equations of motion are

$$\begin{aligned} \mathcal{E}_a^\mu &\equiv e\{R_a{}^\mu(\omega) - \frac{1}{2}e^\mu{}_a R(\omega)\} + \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\bar{\psi}_\nu\gamma_\rho\gamma_a\mathcal{D}_\lambda(\omega)\psi_\rho = 0 \\ \Psi^\mu &\equiv -e\gamma^{\mu\nu\lambda}\mathcal{D}_\nu(\omega)\psi_\lambda = 0. \end{aligned} \quad (3.3)$$

In the absence of auxiliary fields one can also couple Yang-Mills to supergravity [20], and of course that is how it was first done. Since the action changes due to the coupling, so do the transformation rules. The complete action then consists of the sum of (3.2) and the Yang-Mills part

$$\begin{aligned} \mathcal{L}_{\text{YM}} &= -e\beta \text{tr}\{\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\bar{\lambda}\mathcal{D}(\omega, A)\lambda + \frac{1}{8}\bar{\psi}_\mu\gamma^{ab}\gamma^\mu\lambda(F_{ab} + \hat{F}_{ab}) \\ &\quad + \frac{1}{128}\beta\bar{\lambda}\gamma_{abc}\lambda \text{tr}\bar{\lambda}\gamma^{abc}\lambda\}. \end{aligned} \quad (3.4)$$

The action $\mathcal{L}_{\text{SG}} + \mathcal{L}_{\text{YM}}$ is invariant under the following transformation rules for the Yang-Mills fields

$$\delta A_\mu = \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda \quad \delta\lambda = -\frac{1}{4}\gamma^{ab}\epsilon\hat{F}_{ab}(A) \quad (3.5)$$

while the transformation rules of the supergravity fields receive $O(\beta)$ modifications:

$$\delta_\beta\psi_\mu = \frac{1}{32}\beta(e_\mu{}^a\gamma^{bc} - \frac{2}{3}\gamma_\mu{}^{abc})\epsilon \text{tr}\bar{\lambda}\gamma_{abc}\lambda. \quad (3.6)$$

Let us now construct an R^2 action using again an analogy between supergravity and Yang-Mills, similar to the one we employed in section 2.

Consider an $\text{SO}(3, 1)$ Yang-Mills multiplet $[\Omega_\mu{}^{ab}, \Sigma^{ab}]$. The corresponding $\text{SO}(3, 1)$ supersymmetric Yang-Mills action

$$\begin{aligned} \mathcal{L}_2 &= -e\alpha\{\frac{1}{4}R_{\mu\nu}{}^{ab}(\Omega)R^{\mu\nu ab}(\Omega) + \frac{1}{2}\bar{\Sigma}^{ab}\mathcal{D}(\omega, \Omega)\Sigma_{ab} \\ &\quad + \frac{1}{8}\bar{\psi}_\mu\gamma^{cd}\gamma^\mu\Sigma_{ab}(R_{cd}{}^{ab}(\Omega) + \hat{R}_{cd}{}^{ab}(\Omega)) + \frac{1}{128}\alpha\bar{\Sigma}^{ab}\gamma^{pqr}\Sigma_{ab}\bar{\Sigma}^{cd}\gamma_{pqr}\Sigma_{cd}\} \end{aligned} \quad (3.7)$$

is invariant under the transformations (3.1) with (3.6), and (3.5) with the replacement $[A_\mu, \lambda] \rightarrow [\Omega_\mu^{ab}, \Sigma^{ab}]$ and $\beta \rightarrow \alpha$. If we now replace in (3.7) $[\Omega_\mu^{ab}, \Sigma^{ab}]$ by $[\omega_\mu^{ab}, \psi^{ab}]^\dagger$ we lose invariance. The reason is that the transformation rules of ω_μ^{ab} and ψ^{ab} do not have the correct form (3.5):

$$\delta\omega_\mu^{ab}(e, \psi) = \frac{1}{2}\bar{\epsilon}\gamma_\mu\psi^{ab} - \frac{3}{4}\bar{\epsilon}\gamma^{[a}\psi^{bc]}e_{\mu c} \quad (3.8)$$

$$\delta\psi_{ab} = -\frac{1}{4}\gamma^{cd}\epsilon\hat{R}_{cdab}(\omega). \quad (3.9)$$

In (3.8) the difference is due to the last term. The reason that (3.9) does not quite have the structure of (3.5) is that the supercovariant curvature \hat{R} contains covariantisations for the complete transformation (3.8), instead of for the first term only.

The error is of $O(\alpha^0)$ in the variation of ω_μ^{ab} and ψ^{ab} , and of $O(\alpha)$ in the variation of the action. Only those ω_μ^{ab} and ψ^{ab} which replaced Ω and Σ in (3.7) contribute to the non-invariance. Therefore the damage is limited. Furthermore, the unwanted $O(\alpha^0)$ variations in ω_μ^{ab} and ψ^{ab} are proportional to equations of motion of the R action (3.2). Indeed, the gravitino equation of motion Ψ_a (3.3) implies:

$$\begin{aligned} \gamma^{ab}\psi_{ab} &= -e^{-1}\gamma^a\Psi_a \\ \gamma^b\psi_{ab} &= e^{-1}[\Psi_a - \frac{1}{2}\gamma_a\gamma^b\Psi_b] \\ \gamma_{[a}\psi_{bc]} &= \frac{1}{3}e^{-1}\epsilon_{abcd}\gamma_5\Psi^d. \end{aligned} \quad (3.10)$$

Therefore invariance can be restored by suitably modifying the transformation rule of ψ_μ . This works to $O(\alpha)$ only, since the modification (3.6), and these new modifications, cause new $O(\alpha)$ variations of ω_μ^{ab} and ψ^{ab} , which violate invariance of the action, etc.

Let us work out this procedure in more detail. The variation of (3.7) due to arbitrary transformations of Ω and Σ takes the form:

$$\delta\mathcal{L}_2 = \alpha(\delta\Omega_\mu^{ab}\mathcal{L}_{ab}^\mu + \delta\Sigma^{ab}\mathcal{J}_{ab}) \quad (3.11)$$

where

$$\mathcal{L}^{\mu ab} = \mathcal{D}_\nu(\Omega)(eR^{\nu\mu ab}(\Omega) + \frac{1}{2}e\bar{\psi}_\lambda\gamma^{\nu\mu}\gamma^\lambda\Sigma^{ab}) + e\bar{\Sigma}^{ac}\gamma^\mu\Sigma^b_c \quad (3.12)$$

$$\mathcal{J}^{ab} = -e\{\mathcal{D}(\omega, \Omega)\Sigma^{ab} + \frac{1}{4}\gamma^\mu\gamma^{\lambda\rho}\psi_\mu(R_{\lambda\rho}^{ab}(\Omega) - \bar{\psi}_\lambda\gamma_\rho\Sigma^{ab})\}. \quad (3.13)$$

Here we have neglected terms in the variation which are $O(\alpha^2)$. Of course (3.12) and (3.13) are the Yang-Mills equations of motion for Ω and Σ . Now we replace in (3.11)–(3.13) Ω by ω and Σ by ψ . Then (3.11) gives the variation of the action $\mathcal{L} = \mathcal{L}_{SG} + \mathcal{L}_2$, if we take for $\delta\Omega$ and $\delta\Sigma$ the following variations:

$$\begin{aligned} \delta\Omega_\mu^{ab} &= -\frac{3}{4}\bar{\epsilon}\gamma^{[a}\psi^{bc]}e_{\mu c} \\ &= -\frac{1}{4}e^{-1}\epsilon_{\mu abc}\bar{\epsilon}\gamma_5\Psi^c \end{aligned} \quad (3.14)$$

$$\begin{aligned} \delta\Sigma^{ab} &= -\frac{1}{4}\gamma_{cd}\epsilon(\frac{3}{2}\bar{\psi}^c\gamma^{[a}\psi^{bd]}) \\ &= \frac{1}{8}e^{-1}\epsilon^{abcf}\gamma_{cd}\epsilon\bar{\psi}^d\gamma_5\Psi_f \end{aligned} \quad (3.15)$$

which are the deviations from the Yang-Mills transformation rules. This implies that $\delta\mathcal{L}$ can be written as

$$\delta\mathcal{L} = \frac{1}{4}\alpha e^{-1}\epsilon_{\mu abc}\{\bar{\epsilon}\gamma_5\mathcal{L}^{cab} + \frac{1}{2}\bar{\epsilon}\gamma^{cd}\mathcal{J}^{ab}\bar{\psi}_d\gamma_5\}\Psi^\mu \quad (3.16)$$

† Here the explicit form of the gravitino curvature is $\psi_{\mu\nu} \equiv \mathcal{D}_\mu(\omega)\psi_\nu - \mathcal{D}_\nu(\omega)\psi_\mu$.

which can be cancelled by the following additional transformation of the gravitino:

$$\delta\psi_\mu = -\frac{1}{4}\alpha e^{-1}\epsilon_{\mu abc}\{\gamma_5\mathcal{E}^{cab} + \frac{1}{2}\gamma_5\psi_d\bar{e}\gamma^{cd}\mathcal{G}^{ab}\}. \quad (3.17)$$

Thus invariance of the action, at least to $O(\alpha)$, is restored.

The procedure outlined above has the disadvantage that it cannot be immediately generalised to higher orders in α , since it depends crucially on the fact that the extra transformations (3.14) and (3.15) were equations of motions themselves. This also implies that the result is valid in four dimensions only, since the last relation in (3.10), which we used in (3.14)–(3.15), is specific for $d=4$.

An alternative procedure is to use the fact that the Yang–Mills equations of motion (3.12) and (3.13), with the substitution $\Omega \rightarrow \omega$ and $\Sigma \rightarrow \psi$, can be rewritten in terms of the equations of motion (3.3). This was shown for ten dimensions in [6], neglecting quartic fermions in the action. Ultimately this result is a consequence of the Gauss–Bonnet theorem (see (2.29)).

Although this second approach leads to more complicated transformation rules, it has the advantage that it can be generalised in principle to higher orders in α , since it holds for *arbitrary* variations of ω_μ^{ab} and ψ^{ab} . An interesting feature of this second approach is that it leads to different transformation rules to the ones found above. In particular, also the transformation rule of the vierbein has to be modified by $O(\alpha)$ terms.

Let us now present the result of this second approach. First we consider (3.12) (with the replacement $\Omega \rightarrow \omega$ and $\Sigma \rightarrow \psi$). A lengthy calculation gives:

$$\begin{aligned} \mathcal{E}^\mu_{ab} = & 2ee^{\mu c}D_{[a}\hat{R}_{b]c} + \frac{1}{2}e\bar{\psi}_\nu\gamma^{\mu\nu}\mathcal{D}\psi_{ab} + \frac{3}{2}e\bar{\psi}_\nu\gamma_{[a}(D^{[\nu}\psi_{b c]})e^{\mu]c} - \frac{1}{2}\mathcal{D}_\lambda\{\bar{\psi}^{[\lambda}\gamma^{\mu]}_{abc}\Psi^c\} \\ & + e^{\mu d}\{\frac{1}{2}\bar{\psi}_{ab}\Psi_d + \frac{1}{4}\bar{\psi}_{ab}\gamma_c\gamma_d\Psi^c - \frac{1}{2}\bar{\psi}_{[a}\gamma^c\gamma_{b]}\Psi_c - \frac{1}{2}\bar{\psi}^c_{[a}\gamma_{b]cde}\Psi^e\}. \end{aligned} \quad (3.18)$$

All terms in (3.18) are proportional to equations of motion. The supercovariant Ricci tensor can be rewritten in the form

$$\hat{R}_{\mu a}(\omega) = e^{-1}\{\mathcal{E}_{\mu a} - \frac{1}{2}e_{\mu a}\mathcal{E} + \frac{1}{4}\bar{\psi}_\nu\gamma_{\mu a}\Psi^\nu + \frac{1}{4}\bar{\psi}_\nu\gamma_\mu{}^\nu\Psi_a + \frac{1}{4}\bar{\psi}_\nu\gamma_a{}^\nu\Psi_\mu + \frac{1}{4}e_{\mu a}\bar{\psi}_\nu\gamma^{\nu\rho}\Psi_\rho\}. \quad (3.19)$$

Using the Bianchi identity for ψ_{ab} we can rewrite $\mathcal{D}\psi_{ab}$ in terms of equations of motion as well. The result is

$$\mathcal{D}\psi_{ab} = 2e^\nu_{[a}\mathcal{D}_{\nu]}(\omega)\{e^{-1}(\Psi_{b]} - \frac{1}{2}\gamma_{b]}\gamma_c\Psi^c)\} + \gamma_\nu\psi_{[a}\hat{R}^\nu_{b]}(\omega). \quad (3.20)$$

Therefore (3.12) can be completely expressed in terms of equations of motion. We can rewrite (3.13) in the form

$$\mathcal{G}^{ab} = -e\mathcal{D}\psi^{ab} + \frac{1}{8}\epsilon^{abcd}\gamma^\mu\gamma^\nu{}_c\psi_\mu\bar{\psi}_\nu\gamma_5\Psi_d \quad (3.21)$$

which is also a combination of equations of motion of the supergravity action (3.2). Therefore the variation of the action (3.16) can also be cancelled by changing the transformation rules of both the vierbein and the gravitino. In calculating the modifications to the transformation rules, the first term in (3.18) does not contribute, since the antisymmetric part of the Ricci tensor vanishes. Otherwise no significant simplifications occur. We shall refrain from presenting the explicit form of these additional transformation rules in this second approach.

To end this section, let us mention briefly the old minimal system of auxiliary fields [15]. In that formulation, the auxiliary fields are a scalar S , a pseudoscalar P , and a 4-vector V_a . In this formulation, we cannot find a combination of spin-connection and auxiliary fields which transforms exactly as the Yang–Mills gauge field. Therefore the Yang–Mills analogy used in section 2 fails. Of course it is possible to use the

method of this section to obtain an $O(\alpha)$ invariant with the old minimal fields. Since the complete result is known [16], we will not discuss this possibility further.

4. Supersymmetric Chern–Simons terms

In order to find supersymmetric actions with Lorentz Chern–Simons terms in $d = 4$, $N = 1$ supergravity we again make use of the supergravity Yang–Mills analogy. Here we present only the results of the calculation using the new minimal formulation. First we discuss the construction of actions containing Yang–Mills Chern–Simons terms, and then the analogy will give us the related result for the Lorentz group. We also discuss how the structure changes when chiral multiplets are included. After a duality transformation the resulting action has the same structure as the effective action obtained in [8] from ten dimensions.

The pure Chern–Simons action for $d = 4$, $N = 1$ super–Yang–Mills is based on the G component of a chiral multiplet with weight two (2.6). This action is given by:

$$\mathcal{L}_G = e\{G + \frac{1}{2}i\bar{\psi}_\mu\gamma_5\gamma^\mu\phi + \frac{1}{4}i\bar{\psi}_\mu\gamma_5\gamma^{\mu\nu}(A + i\gamma_5 B)\psi_\nu\}. \quad (4.1)$$

Using (2.14) we find immediately the supersymmetric Chern–Simons action for a Yang–Mills multiplet

$$\begin{aligned} \mathcal{L}_{CS} &= -e\beta \operatorname{tr}\{\frac{1}{8}i\varepsilon^{abcd}\hat{F}_{ab}\hat{F}_{cd} - \frac{1}{2}i\bar{\lambda}\gamma_5\mathcal{D}\lambda - \frac{1}{4}i\bar{\psi}_\mu\gamma^\mu(\frac{1}{2}\gamma_5\gamma^{ab}\lambda\hat{F}_{ab} - \lambda D) \\ &\quad + \frac{1}{16}i\bar{\psi}_\mu\gamma^{\mu\nu}(\gamma_5\bar{\lambda}\lambda - \bar{\lambda}\gamma_5\lambda)\psi_\nu\} \\ &= -\frac{1}{4}i\beta\partial_\mu \operatorname{tr}\{\varepsilon^{\mu\nu\rho\sigma}(A_\nu\partial_\rho A_\sigma - \frac{1}{3}A_\nu A_\rho A_\sigma) - e\bar{\lambda}\gamma_5\gamma^\mu\lambda\} \end{aligned} \quad (4.2)$$

where A_μ is the gauge field of the Yang–Mills multiplet (2.7). It is now easy to find the analogous expression for the Lorentz Chern–Simons term. Since (2.10) transforms as an $SO(3, 1)$ Yang–Mills multiplet (see (2.19)), one can make the replacements

$$\begin{aligned} A_\mu &\leftrightarrow \Omega_\mu^{ab} & \chi &\leftrightarrow \psi^{ab} \\ D &\leftrightarrow -2\hat{F}^{ab}(V_+) & \hat{F}_{\mu\nu}(A) &\leftrightarrow \hat{R}_{\mu\nu}^{ab}(\Omega^-) \end{aligned} \quad (4.3)$$

in the Lagrangian (4.2). Working out explicitly the Yang–Mills trace for the group $SO(3, 1)$ we obtain

$$\mathcal{L}_{LCS} = -\frac{1}{4}i\alpha\partial_\mu\{\varepsilon^{\mu\nu\rho\sigma}(\Omega_{\nu-}^{ab}\partial_\rho\Omega_{\sigma-}^{ab} - \frac{2}{3}\Omega_{\nu-}^{ab}\Omega_{\rho-}^{ac}\Omega_{\sigma-}^{cb}) - e\bar{\psi}^{ab}\gamma_5\gamma^\mu\psi_{ab}\} \quad (4.4)$$

with an arbitrary coupling constant α . This action is the supersymmetric extension of the Lorentz Chern–Simons term in the context of new minimal supergravity.

Let us now use the actions (4.2) and (4.4) as ingredients in the coupling of a chiral multiplet of weight zero to Yang–Mills and to new minimal supergravity. We shall see that the action of this coupled system enables us to make contact with an effective action resulting from string compactification [8].

Local tensor calculus implies that we can multiply two chiral multiplets of weight n and m to form a third one of weight $n + m$:

$$[A_1, B_1, \phi_1, F_1, G_1]_n \otimes [A_2, B_2, \phi_2, F_2, G_2]_m = [A_3, B_3, \phi_3, F_3, G_3]_{n+m} \quad (4.5)$$

where

$$\begin{aligned}
 A_3 &= A_1 A_2 - B_1 B_2 & B_3 &= A_1 B_2 + A_2 B_1 \\
 \phi_3 &= (A_1 - i\gamma_5 B_1)\phi_2 + (A_2 - i\gamma_5 B_2)\phi_1 \\
 F_3 &= A_1 F_2 + A_2 F_1 + B_2 G_1 + B_1 G_2 - \bar{\phi}_1 \phi_2 \\
 G_3 &= A_1 G_2 + A_2 G_1 - B_2 F_1 - B_1 F_2 + i\bar{\phi}_1 \gamma_5 \phi_2.
 \end{aligned} \tag{4.6}$$

In order to construct actions we need $n+m=2$, here we choose $n=0$, $m=2$. Using a chiral multiplet of weight zero for the '1'-multiplet, (2.14) for the '2'-multiplet and the F -type action (2.9) we obtain

$$\begin{aligned}
 \mathcal{L}_{B-YM} &= e\{F_3 + \frac{1}{2}\bar{\psi}_\mu \gamma^\mu \phi_3 + \frac{1}{4}\bar{\psi}_\mu \gamma^{\mu\nu} (A_3 + i\gamma_5 B_3)\psi_\nu\} \\
 &= A\mathcal{L}_{YM} + B\mathcal{L}_{CS} - e\beta \operatorname{tr}[\frac{1}{4}\bar{\lambda}\lambda (F + \frac{1}{2}\bar{\psi}_\mu \gamma^\mu \phi) + \frac{1}{4}i\bar{\lambda}\gamma_5 \lambda (G - \frac{1}{2}i\bar{\psi}_\mu \gamma^\mu \gamma_5 \phi) \\
 &\quad - \frac{1}{4}\bar{\phi}\gamma^{ab}\lambda\hat{F}_{ab}(A) - \frac{1}{2}i\bar{\phi}\gamma_5 \lambda D].
 \end{aligned} \tag{4.7}$$

Here \mathcal{L}_{YM} and \mathcal{L}_{CS} denote the Lagrangians (2.16) and (4.2) respectively. The action (4.7) describes the coupling of a chiral multiplet of weight zero $[A, B, \phi, F, G]_0$ to a Yang-Mills multiplet $[A_\mu, \lambda, D]$. It is important to note that the Lagrangian (4.7) depends on the pseudoscalar B only via $\partial_\mu B$, after a partial integration in the $B\mathcal{L}_{CS}$ term.

In order to write down a complete action for the coupled system we want to consider, we need in addition to (4.7) a contribution that involves kinetic terms for the chiral multiplet. It is straightforward to show that if

$$[A, B, \phi, F, G]_0 \tag{4.8}$$

is a chiral multiplet of weight zero, then

$$\begin{aligned}
 &[F, G, \mathcal{D}(\omega, V)\phi - \frac{1}{12}\gamma^{abc}\phi\hat{H}_{abc} \\
 &\square A - \frac{1}{3}i\epsilon^{abcd}\hat{H}_{bcd}D_a B - \frac{1}{4}\bar{\psi}_{\mu\nu}\gamma^{\mu\nu}\phi \\
 &\square B + \frac{1}{3}i\epsilon^{abcd}\hat{H}_{bcd}D_a A - \frac{1}{4}i\bar{\psi}_{\mu\nu}\gamma^{\mu\nu}\gamma_5 \phi]
 \end{aligned} \tag{4.9}$$

(where \square denotes the supercovariant d'Alembert operator) is a chiral multiplet of weight two, the associated kinetic multiplet. Again making use of (4.6), where the '1'-multiplet is given by (4.8) and the '2'-multiplet by (4.9), and by using the F -type action (2.9) we find an action that involves kinetic terms for the multiplet (4.8):

$$\begin{aligned}
 \mathcal{L}_{B-kin} &= e\{A[\square A - \frac{1}{3}i\epsilon^{abcd}\hat{H}_{bcd}D_a B - \frac{1}{4}\bar{\psi}_{\mu\nu}\gamma^{\mu\nu}\phi] \\
 &\quad + B[\square B + \frac{1}{3}i\epsilon^{abcd}\hat{H}_{bcd}D_a A - \frac{1}{4}i\bar{\psi}_{\mu\nu}\gamma^{\mu\nu}\gamma_5 \phi] + F^2 + G^2 \\
 &\quad - \bar{\phi}\mathcal{D}(\omega, V)\phi + \frac{1}{12}\bar{\phi}\gamma^{abc}\phi\hat{H}_{abc} \\
 &\quad + \frac{1}{2}\bar{\psi}_\mu \gamma^\mu [(A - i\gamma_5 B)(\mathcal{D}(\omega, V)\phi - \frac{1}{12}\gamma^{abc}\phi\hat{H}_{abc}) + (F - i\gamma_5 G)\phi] \\
 &\quad + \frac{1}{4}\bar{\psi}_\mu \gamma^{\mu\nu}\psi_\nu (AF - BG) + \frac{1}{4}i\bar{\psi}_\mu \gamma_5 \gamma^{\mu\nu}\psi_\nu (AG + FB)\}.
 \end{aligned} \tag{4.10}$$

By removing supercovariantisations and performing partial integrations, (4.10) can be simplified considerably:

$$\begin{aligned}
 \mathcal{L}_{B-kin} &= e\{\partial_\mu A[-\partial^\mu A - \bar{\phi}\gamma^{\mu\nu}\psi_\nu + \bar{\phi}\psi^\mu] \\
 &\quad + \partial_\mu B[-\partial^\mu B - i\bar{\phi}\gamma_5 \gamma^{\mu\nu}\psi_\nu + i\bar{\phi}\gamma_5 \psi^\mu - \frac{2}{3}ie^{-1}\epsilon^{\mu\nu\rho\sigma}(\partial_\nu B_{\rho\sigma})A] \\
 &\quad + F^2 + G^2 - \bar{\phi}\mathcal{D}(\omega, V)\phi + \frac{1}{12}\bar{\phi}\gamma^{abc}\phi\hat{H}_{abc} + \frac{1}{8}\bar{\phi}\gamma^{\mu\nu\rho}\phi\bar{\psi}_\mu \gamma_\nu \psi_\rho \\
 &\quad + \frac{1}{8}\bar{\phi}\gamma^\mu \gamma_5 \phi\bar{\psi}_\nu \gamma_5 \gamma_\mu \psi^\nu\}.
 \end{aligned} \tag{4.11}$$

(Here $B_{\rho\sigma}$ denotes the auxiliary field of the supergravity multiplet.) As was the case in (4.7), the action (4.11) depends on the pseudoscalar B only via $\partial_\mu B$.

Having constructed the action that describes the coupling of the chiral multiplet (4.8) to Yang-Mills and the kinetic action for (4.8), it remains to include the Lorentz Chern-Simons term and the associated R^2 term. This is achieved by again making use of the Yang-Mills supergravity analogy. With (4.3) and (4.7) we immediately get:

$$\begin{aligned} \mathcal{L}_{\text{B-SG}} = & A\mathcal{L}_2 + B\mathcal{L}_{\text{LCS}} - e\alpha\{\frac{1}{4}\bar{\psi}^{ab}\psi_{ab}(F + \frac{1}{2}\bar{\psi}_\mu\gamma^\mu\phi) + \frac{1}{4}i\bar{\psi}^{ab}\gamma_5\psi_{ab}(G - \frac{1}{2}i\bar{\psi}_\mu\gamma^\mu\gamma_5\phi) \\ & - \frac{1}{4}\bar{\phi}\gamma^{\mu\nu}\psi_{ab}\hat{R}_{\mu\nu}{}^{ab}(\Omega^-) + i\bar{\phi}\gamma_5\psi_{ab}\hat{F}^{ab}(V_+)\}. \end{aligned} \quad (4.12)$$

Here \mathcal{L}_2 denotes (2.21) (containing an R^2 term) and \mathcal{L}_{LCS} is the Lorentz Chern-Simons action (4.4).

Now we are able to write down the complete action describing the coupling of a chiral multiplet of weight zero to Yang-Mills and to new minimal supergravity:

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{SG}} + \mathcal{L}_{\text{B-kin}} + \mathcal{L}_{\text{B-YM}} + \mathcal{L}_{\text{B-SG}} \quad (4.13)$$

where \mathcal{L}_{SG} is the supergravity R action (2.13) involving kinetic terms for the supergravity fields. Since (4.13) and the transformation rules of the chiral multiplet depend only on the derivative of the pseudoscalar field B , we can now perform a duality transformation [21], which will replace B by an antisymmetric tensor gauge field $A_{\mu\nu}$ (see also [13, 22]).

The $\partial_\mu B$ contribution to (4.13) can be written:

$$\mathcal{L}_{\partial B} = -e\partial_\mu B\partial^\mu B + 2i\partial_\mu B\epsilon^{\mu\nu\lambda\rho}C_{\nu\lambda\rho} \quad (4.14)$$

where

$$\begin{aligned} \epsilon_{\mu\nu\lambda\rho}C^{\nu\lambda\rho} = & \epsilon_{\mu\nu\lambda\rho}\{\frac{1}{8}\beta[(X^{\text{CS}})^{\nu\lambda\rho} + \frac{1}{6}\text{tr}\bar{\lambda}\gamma^{\nu\lambda\rho}\lambda] + \frac{1}{8}\alpha[(X^{\text{LCS}})^{\nu\lambda\rho} + \frac{1}{6}\bar{\psi}^{ab}\gamma^{\nu\lambda\rho}\psi_{ab}] \\ & + \frac{1}{4}\bar{\phi}\gamma^{\nu\lambda}\psi^\rho - \frac{1}{12}\bar{\phi}\gamma^{\nu\lambda\rho\sigma}\psi_\sigma - \frac{1}{3}(\partial^\nu B^{\lambda\rho})A\}. \end{aligned} \quad (4.15)$$

$X_{\mu\nu\lambda}^{\text{CS}}$ is the Yang-Mills Chern-Simons term

$$X_{\mu\nu\lambda}^{\text{CS}} \equiv \text{tr}[A_{[\mu}\partial_\nu A_{\lambda]} - \frac{1}{3}A_{[\mu}A_\nu A_{\lambda]}] \quad (4.16)$$

and $X_{\mu\nu\lambda}^{\text{LCS}}$ is the corresponding Lorentz Chern-Simons term.

We now replace $\partial_\mu B$ by a vector field t_μ in the action and all transformation rules. To ensure that the result is equivalent to (4.14), we add a term containing a Lagrange multiplier field $A_{\mu\nu}$. The resulting action reads

$$\mathcal{L}_{\partial B} = -et_\mu t^\mu - 2it_\mu\epsilon^{\mu\nu\lambda\rho}C_{\nu\lambda\rho} - i\epsilon^{\mu\nu\lambda\rho}A_{\lambda\rho}\partial_\mu t_\nu. \quad (4.17)$$

The equation of motion of the field t_μ yields

$$t^\mu = -\frac{1}{2}ie^{-1}\epsilon^{\mu\nu\lambda\rho}\{\partial_\nu A_{\lambda\rho} - 2C_{\nu\lambda\rho}\}. \quad (4.18)$$

Using this to eliminate t^μ we obtain the following action

$$\mathcal{L}_{\partial B} = -\frac{3}{2}eF'_{\mu\nu\lambda}F'^{\mu\nu\lambda} \quad (4.19)$$

where

$$F'_{\mu\nu\lambda} \equiv F_{\mu\nu\lambda} - 2C_{\mu\nu\lambda} \quad F_{\mu\nu\lambda} \equiv \partial_{[\mu}A_{\nu\lambda]}. \quad (4.20)$$

The new action \mathcal{L}_{tot} is obtained by replacing in (4.13) the terms given in (4.14) by the result of the duality transformation (4.19).

The duality transformation preserves supersymmetry. The new transformation rules are obtained as follows [21]. First one eliminates from \mathcal{L}_{tot} the auxiliary fields of the chiral multiplet, F and G . The number of off-shell degrees of freedom changes by the duality transformation, and eliminating F and G compensates for this. Then we assign to t_μ the transformation rule $\delta t_\mu = \partial_\mu \delta B$. Consider now the action \mathcal{L}_{tot} in the intermediate form where $\mathcal{L}_{\partial B}$ is given by (4.17). This action is invariant, except for variations containing $\partial_{[\mu} t_{\nu]}$. These terms do not vanish because t_μ is unconstrained. They are cancelled instead by a proper choice of $\delta A_{\mu\nu}$. The result is

$$\delta A_{\mu\nu} = -\frac{1}{2}\bar{\epsilon}\gamma_{\mu\nu}\phi - \frac{2}{3}A\delta B_{\mu\nu} - \frac{1}{4}\beta \text{tr} A_{[\mu}\delta A_{\nu]} - \frac{1}{4}\alpha\Omega_{[\mu}{}^{ab}\delta\Omega_{\nu]}{}^{ab}. \quad (4.21)$$

In fact, this result preserves off-shell supersymmetry. The chiral multiplet (4.8) has been replaced by an off-shell linear multiplet [19]

$$[A, A_{\mu\nu}, \phi]. \quad (4.22)$$

This multiplet is in lowest order in α and β the same as (2.23), except that the constraint (2.24) for the vector v_a has been solved in terms of $A_{\mu\nu}$.

The Lagrangian (4.19) and \mathcal{L}_{tot} are invariant under the gauge transformations

$$\delta A_{\mu\nu} = \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu. \quad (4.23)$$

Invariance under Yang–Mills and Lorentz transformations requires additional transformations of $A_{\mu\nu}$. Because of the presence of the Chern–Simons terms in $C^{\nu\lambda\rho}$ (4.15), we have under infinitesimal Yang–Mills and Lorentz transformations

$$\delta C_{\mu\nu\lambda} = \frac{1}{8}\beta \text{tr} \partial_{[\mu}(\Lambda\partial_\nu A_{\lambda]}) + \frac{1}{8}\alpha\partial_{[\mu}(\Lambda^{ab}\partial_\nu\Omega_{\lambda]}{}^{ab}). \quad (4.24)$$

Hence, with the following additional gauge transformation of the tensor field $A_{\mu\nu}$:

$$\delta A_{\mu\nu} = \frac{1}{4}\beta \text{tr} \Lambda\partial_{[\mu}A_{\nu]} + \frac{1}{4}\alpha\Lambda^{ab}\partial_{[\mu}\Omega_{\nu]}{}^{ab} \quad (4.25)$$

the field strength $F'_{\mu\nu\lambda}(A)$, as well as the complete action, is invariant under Yang–Mills and Lorentz transformations. We note that a similar mechanism (i.e. the combination of a two-index tensor gauge field with the Chern–Simons term in order to maintain gauge invariance) is essential to couple Maxwell [2] or Yang–Mills [3] to supergravity in ten dimensions. The addition of the Lorentz Chern–Simons form in $d=10$ is essential for the cancellation of anomalies, but breaks supersymmetry.

Now we can compare the total action (4.13), where the ∂B part (i.e. (4.14)) is replaced by (4.19), with an action derived in [8]. The critical point in our derivation of (4.19) is the possibility to perform a duality transformation in (4.13) in order to get rid of the pseudoscalar B . The duality transformation necessitates the introduction of an antisymmetric tensor gauge field $A_{\mu\nu}$. In the total action it appears only through its field strength, $F'_{\mu\nu\lambda}$, which also contains Chern–Simons terms. The same structure can be found in [8], where an effective action resulting from dimensional reduction from $N=1$, $d=10$ supergravity to four dimensions is derived. The field A of the chiral multiplet mimicks the role of the scalars ϕ and σ appearing in [8]. The scalar ϕ belongs to the ten-dimensional supergravity multiplet, and σ is related to a rescaling of the original ten-dimensional metric.

Hence we demonstrated that in the framework of new minimal supergravity coupled to a chiral multiplet and to Yang–Mills, it is possible to construct an action that resembles the Witten action [8]. The essential step in the derivation of this action, which permits us to include the Lorentz Chern–Simons term, is again the analogy between Yang–Mills and new minimal supergravity. A crucial point in this derivation is the fact that all B -dependent contributions can be written in terms of $\partial_\mu B$. This

allows us to perform the required duality transformation. In these calculations the auxiliary field structure of new minimal supergravity (i.e. the \hat{H} field) plays an essential role.

The results of this section can also be obtained for on-shell supergravity by the procedure of section 3. To do so one constructs the action \mathcal{L}_{tot} , as in (4.13), for the Yang-Mills sector alone (i.e. $\alpha = 0$). After the duality transformation one eliminates all the auxiliary fields. This gives the on-shell version of \mathcal{L}_{tot} , with the corresponding on-shell transformation rules. Finally, as in section 3, one replaces $[A_\mu, \lambda]$ by $[\omega_\mu^{ab}, \psi^{ab}]$. The resulting action is not quite supersymmetric, but in the same way as in section 3 supersymmetry to $O(\alpha)$ is restored by a modification of the transformation rule of the gravitino.

5. Conclusions

In $d = 4$, $N = 1$ new minimal supergravity it is possible to combine the spin-connection and the supercovariant curvature of an auxiliary antisymmetric tensor gauge field in such a way that the resulting quantity transforms as an $\text{SO}(3, 1)$ Yang-Mills gauge field under local supersymmetry. This permits us to use the locally supersymmetric invariant Yang-Mills action to derive an *invariant* R^2 action for new minimal supergravity. The same procedure allows a simple derivation of supersymmetric Chern-Simons terms. In this way we obtain actions of which the general features are similar to effective actions derived from $d = 10$.

It turns out that even in an on-shell construction of an R^2 action the interpretation of the spin-connection as an $\text{SO}(3, 1)$ Yang-Mills gauge field is helpful. Here this analogy leads to an R^2 action that is invariant under supersymmetry to $O(\alpha)$, where α is the inverse string tension. In addition, this procedure requires α -dependent modifications of the transformation rules of (at least) the gravitino. In principle this procedure can be extended to higher orders in α .

It is clear that this analogy between supergravity and Yang-Mills theory works best if the supergravity theory contains an antisymmetric tensor gauge field. This can be understood as follows. The supersymmetry transformation of $\omega_\mu^{ab}(e, \psi)$ in the absence of auxiliary fields reads:

$$\delta \omega_\mu^{ab}(e, \psi) = \frac{1}{2} \bar{\epsilon} \gamma_\mu \psi^{ab} - \frac{3}{4} \bar{\epsilon} \gamma^{[a} \psi^{bc]} e_\mu^c. \quad (5.1)$$

The second term prevents the interpretation of ω_μ^{ab} itself as the Yang-Mills field. To cancel this term, we need another field, say $H_{\mu ab}$, transforming into $\bar{\epsilon} \gamma_{[a} \psi_{bc]} e_\mu^c$. Because ψ_{ab} is a curvature, it satisfies a Bianchi identity, so that we can restrict ourselves to tensors H satisfying $D_{[a} H_{bcd]} = 0$. This constraint is solved by setting H equal to the curvature of an antisymmetric tensor gauge field $B_{\mu\nu}$. Thus our requirement that the transformation of a Yang-Mills field should appear, uniquely leads to the presence of an antisymmetric tensor, whose supersymmetry transformation rule can be evaluated from the above.

This does not mean that our construction in section 2 would also have worked in the absence of the second auxiliary field of the new minimal formalism, V_μ . One can see this by considering the equations of motion of the auxiliary fields V_μ and $B_{\mu\nu}$. They read, respectively,

$$\hat{H}_{\mu\nu\lambda} = 0 \quad (5.2)$$

$$\partial_\mu \{ e \hat{H}^{\mu\nu\lambda} \} + \frac{1}{3} i \epsilon^{\nu\lambda\mu\rho} F_{\mu\rho}(V) = 0. \quad (5.3)$$

These equations do not imply that V_μ vanishes, and therefore its presence cannot be simply ignored.

The equations of motion (5.2) and (5.3) can be used to eliminate the auxiliary fields from the supergravity action (2.13), which then becomes equal to the on-shell action (3.2). In the $R + R^2$ actions of sections 2 and 3 such a correspondence does not hold. The auxiliary fields become physical fields, and can no longer be eliminated algebraically. Therefore different off-shell versions of supergravity yield physically different $R + R^2$ actions. Ideally, one should let the string theory decide which version is the correct one. However, the nature of this decision is controversial [23].

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